

# QFT with twisted Poincaré invariance and the Moyal product

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**ABSTRACT:** We study the consequences of twisting the Poincaré invariance in a quantum field theory. First, we construct a Fock space compatible with the twisting and the corresponding creation and annihilation operators. Then, we show that a covariant field linear in creation and annihilation operators does not exist. Relaxing the linearity condition, a covariant field can be determined. We show that it is related to the untwisted field by a unitary transformation and the resulting  $n$ -point functions coincide with the untwisted ones. We also show that invariance under the twisted symmetry can be realized using the covariant field with the usual product or by a non-covariant field with a Moyal product. The resulting  $S$ -matrix elements are shown to coincide with the untwisted ones up to a momenta dependent phase.

**KEYWORDS:** Quantum Groups, Non-Commutative Geometry, Space-Time Symmetries.

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## 1. Introduction

Invariance under the Poincaré group leads to important restrictions on quantum field theory. In the free linear case, it completely determines the theory [1]. It is then of great interest to examine the consequences of deforming this invariance. Many possible deformations have been considered in the past as, for instance, replacing the Poincaré group by (A)dS group or deforming the Lie algebra structure.

Here, we will examine the possibility of keeping unchanged the algebra structure of the Poincaré universal enveloping algebra (UEA) but deforming its coalgebra structure. The latter is crucial in quantum field theory since the coproduct determines the transformation laws of a system of several particles. A simple way to deform the coproduct  $\Delta$  can be obtained from the usual coproduct  $\Delta_0$  and a twist element  $\mathcal{F}$  [2] as

$$\Delta = \mathcal{F} \Delta_0 \mathcal{F}^{-1}, \tag{1.1}$$

the resulting UEA will be called in the following the twisted Poincaré UEA.<sup>1</sup> A consistent example was recently [3, 4] considered and is given by

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu} \mathcal{P}_\mu \otimes \mathcal{P}_\nu}, \tag{1.2}$$

where  $\theta^{\mu\nu}$ , elements of an anti-symmetric matrix, are the deformation parameters. This example arose when considering quantum field theory (QFT) on the noncommutative space [5–7] where coordinates satisfy the commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \tag{1.3}$$

<sup>1</sup>In fact, a theorem by Drinfeld [2] shows that all deformations of the coproduct of a semi-simple Lie algebra are of this form.

It describes the low energy effective theory on D-branes with an external  $B$ -field. By the Weyl-Moyal correspondence, the theory can be reformulated as a field theory on commutative space but the usual product between two fields being replaced by the Moyal  $*$ -product,<sup>2</sup>

$$(f *_{\theta} g)(x) = e^{\frac{i}{2}\partial_x \theta \partial_y} f(x) g(y) |_{x=y} = m(\mathcal{F}^{-1} \triangleright (f \otimes g))(x), \quad (1.4)$$

where  $m(f \otimes g)(x)$  is the usual pointwise product  $f(x)g(x)$ . The existence of the non-invariant matrix  $\theta$  in the Moyal product makes the theory non-covariant under Lorentz transformations.

It was argued [3, 8] (see [9–12, 24–26] for related works) that even though this non-commutative field theory (NCFT) with the Moyal product is not covariant under the Poincaré UEA, it is nevertheless covariant under the twisted Poincaré UEA with  $\mathcal{F}$  given by eq. (1.2). More precisely, it was noticed that the action of an element  $\mathcal{X}$  of the twisted Poincaré UEA on the algebra of functions verifies:

$$\mathcal{X} \triangleright (f *_{\theta} g) = (\mathcal{X}_{(1)} \triangleright f) *_{\theta} (\mathcal{X}_{(2)} \triangleright g), \quad \Delta(\mathcal{X}) = \mathcal{X}_{(1)} \otimes \mathcal{X}_{(2)}. \quad (1.5)$$

On the other hand it was also advocated that in order to implement this new covariance, one should deform also the commutation relations of creation and annihilation operators [13] (see also [14–16] for further discussions) and in that case, it was concluded that there is no physical difference from the commutative QFT [17]. In addition, it was shown that Wightman functions with the Moyal  $*$ -product coincide with the untwisted ones [18]. The field operator used in these works is not covariant under the Poincaré transformations. One of our goals in this letter will be to find a covariant field operator. This operator allows a transparent and manifest implementation of the twisted symmetry without using the Moyal product. We shall show that the  $n$ -point functions are identical to the untwisted ones and that the  $S$ -matrix elements coincide with the usual ones up to a momenta dependent phase.

Our aim is to study the consequences of the covariance of the quantum field under the twisted Poincaré UEA. First, in section 2, we construct the twisted Fock space and the associated creation and annihilation operators,  $a_{\theta}^{\dagger}(p)$  and  $a_{\theta}(p)$ , are determined in section 3. We show that fields linear in  $a_{\theta}^{\dagger}(p)$  and  $a_{\theta}(p)$  cannot be covariant. In particular, replacing in the usual untwisted field operator  $\Phi_0$  the creation and annihilation operators by  $a_{\theta}^{\dagger}(p)$  and  $a_{\theta}(p)$ , we get a field  $\Phi_{nc\theta}$  which is not covariant. This is the field considered in [13, 18]. A covariant field operator  $\Phi_{\theta}$  can however be found by relaxing the linearity requirement. We show that  $n$ -point functions are related by the simple relation:

$$\begin{aligned} \langle \Omega | (\Phi_{\theta} *_{\vartheta} \cdots *_{\vartheta} \Phi_{\theta})(x_1, \dots, x_n) | \Omega \rangle &= \langle \Omega | (\Phi_{nc\theta} *_{\vartheta-\theta} \cdots *_{\vartheta-\theta} \Phi_{nc\theta})(x_1, \dots, x_n) | \Omega \rangle \\ &= \langle \Omega | (\Phi_0 *_{\vartheta} \cdots *_{\vartheta} \Phi_0)(x_1, \dots, x_n) | \Omega \rangle. \end{aligned} \quad (1.6)$$

The covariant  $n$ -point functions are the ones with  $\vartheta = 0$ . This shows that the covariant theory based on twisted Poincaré and the untwisted one have identical correlation functions.

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<sup>2</sup>We use a simplified notation  $v\theta w$  for  $v_{\mu}\theta^{\mu\nu}w_{\nu}$ .

It also shows that twisting the product without using twisted fields, as is usually the case in NCFT, does not lead to a covariant result. We also argue that the implementation of the twisted covariance in an interacting theory can be realized by an interaction Hamiltonian which is local in the field  $\Phi_\theta$  or equivalently non-local in the  $\Phi_{nc\theta}$  field operator. If we insist on having asymptotic states transforming covariantly under the twisted Poincaré symmetry then, as is shown in section 3, the  $S$ -matrix elements are identical to the untwisted ones up to a momenta dependent phase. The appendix contains the proof of some technical results used in the text.

## 2. Twisted Poincaré UEA and its representation

The twisted coproduct yields transformation laws of multi-particle states under the Poincaré group which are modified with respect to the usual ones. In particular, the action of the UEA on the tensor product of identical particles does not commute any more with the permutation of particles, leading to a twisted Fock space. In this section, we shall briefly review the twisted Poincaré UEA representation and use it for the construction of the Fock space.

Let  $\mathcal{U}$  be the UEA of the Poincaré Lie algebra generated by  $\mathcal{P}_\mu$  and  $\mathcal{M}_{\mu\nu}$ . The one-particle Hilbert space  $\mathcal{H}$  of a scalar particle corresponds to the scalar unitary irreducible representation (UIR). Let  $U(\mathcal{X})$  be the corresponding unitary representation of  $\mathcal{X} \in \mathcal{U}$ . A basis of  $\mathcal{H}$  is given by the momentum eigenstates  $|p\rangle$ .

The tensor product representation  $U^{(n)}(\mathcal{X})$  on  $\mathcal{H}^{\otimes n}$ , the tensor product of  $n$  copies of  $\mathcal{H}$ , is obtained from  $U(\mathcal{X})$  and the coassociative coproduct  $\Delta$  as

$$U^{(n)}(\mathcal{X}) = U^{\otimes n}(\Delta^{(n)}(\mathcal{X})) = U(\mathcal{X}_{(1)}) \otimes \cdots \otimes U(\mathcal{X}_{(n)}), \quad (2.1)$$

where the  $n$ -coproduct  $\Delta^{(n)}(\mathcal{X}) = \mathcal{X}_{(1)} \otimes \cdots \otimes \mathcal{X}_{(n)}$  is defined by

$$\Delta^{(n+1)}(\mathcal{X}) = (\Delta \otimes \text{id}^{\otimes(n-1)})(\Delta^{(n)}(\mathcal{X})), \quad \Delta^{(2)} = \Delta. \quad (2.2)$$

The representation  $X$  of  $\mathcal{X}$  on the tensor algebra  $\mathcal{T}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$  is the direct sum of the tensor product representations:

$$X = 1 \oplus U(\mathcal{X}) \bigoplus_{n=2}^{\infty} U^{(n)}(\mathcal{X}), \quad (2.3)$$

where the first part 1 of  $X$  is the trivial representation.

For the untwisted Poincaré UEA, the coproduct  $\Delta_0$  is given by<sup>3</sup>

$$\Delta_0(\mathcal{X}) = \mathcal{X}^{(2)}, \quad \Delta_0^{(n)}(\mathcal{X}) = \mathcal{X}^{(n)}, \quad (2.5)$$

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<sup>3</sup>For an element  $\mathcal{X}$  of  $\mathcal{U}$  and an element  $\mathcal{Y} = \mathcal{Y}_{(1)} \otimes \mathcal{Y}_{(2)}$  of  $\mathcal{U} \otimes \mathcal{U}$ , we define  $\mathcal{X}_i^{(n)}$ ,  $\mathcal{X}^{(n)}$ ,  $\mathcal{Y}_{ij}^{(n)}$  and  $\mathcal{Y}^{(n)}$  which are elements of  $\mathcal{U}^{\otimes n}$  as

$$\begin{aligned} \mathcal{X}_i^{(n)} &= 1 \otimes \cdots \otimes \mathcal{X}^{i^{\text{th}}} \otimes \cdots \otimes 1, & \mathcal{X}^{(n)} &= \sum_{i=1}^n \mathcal{X}_i^{(n)}, \\ \mathcal{Y}_{ij}^{(n)} &= 1 \otimes \cdots \otimes \mathcal{Y}_{(1)}^{i^{\text{th}}} \otimes \cdots \otimes \mathcal{Y}_{(2)}^{j^{\text{th}}} \otimes \cdots \otimes 1, & \mathcal{Y}^{(n)} &= \sum_{1 \leq i < j \leq n} \mathcal{Y}_{ij}^{(n)}. \end{aligned} \quad (2.4)$$

and the resulting tensor algebra representation  $X_0$  of  $\mathcal{X}$  is

$$X_0 = 1 \oplus U(\mathcal{X}) \bigoplus_{n=2}^{\infty} U_0^{(n)}(\mathcal{X}), \quad U_0^{(n)}(\mathcal{X}) = U^{\otimes n}(\Delta_0^{(n)}(\mathcal{X})). \quad (2.6)$$

In order to construct the Fock space we shall need the flip operation  $\pi_{ij}$  on  $\mathcal{U}^{\otimes n}$  given by

$$\pi_{ij}(\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_n) = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_j^{\text{th}} \otimes \cdots \otimes \mathcal{X}_i^{\text{th}} \otimes \cdots \otimes \mathcal{X}_n. \quad (2.7)$$

The n-coproduct  $\Delta_0^{(n)}$  is then invariant under the flip:

$$\pi_{ij}(\Delta_0^{(n)}(\mathcal{X})) = \Delta_0^{(n)}(\mathcal{X}), \quad (2.8)$$

and the tensor product representation  $U^{(n)}(\mathcal{X})$  commutes with the analogously defined flip maps  $\Pi_{ij}$ 's on  $\mathcal{H}^{\otimes n}$ :

$$[\Pi_{ij}, U_0^{(n)}(\mathcal{X})] = 0. \quad (2.9)$$

Therefore,  $\mathcal{H}^{\otimes n}$  is reducible and contains two Fock spaces, the invariant eigenspaces of  $\Pi_{ij}$ . Eigenvalue +1 (totally symmetric case) corresponds to bosons and -1 (totally antisymmetric case) corresponds to fermions. In the following, we shall concentrate on the bosonic Fock space  $\mathcal{F}$ , which is obtained by projecting the tensor algebra  $\mathcal{T}(\mathcal{H})$  on the +1 eigenspace of  $\Pi_{ij}$ :

$$\mathcal{F}_0 = \mathcal{S}_0 \mathcal{T}(\mathcal{H}). \quad (2.10)$$

Here the projector  $\mathcal{S}_0$  is the symmetrization map and n-boson states are thus given by

$$|p_1, \dots, p_n\rangle_0 = \mathcal{S}_0 |p_1\rangle \otimes \cdots \otimes |p_n\rangle, \quad (2.11)$$

and their scalar product is given by

$${}_0\langle p_1, \dots, p_n | q_1, \dots, q_n \rangle_0 = \sum_{\mathcal{P}} \delta(p_1 - q_{\mathcal{P}1}) \cdots \delta(p_n - q_{\mathcal{P}n}), \quad (2.12)$$

where the sum  $\sum_{\mathcal{P}}$  is over all permutations.

We now turn to the twisted Poincaré UEA which has the same algebra structure as the untwisted one but is equipped with a twisted coproduct  $\Delta_{\theta}$ :

$$\Delta_{\theta}(\mathcal{X}) = \mathcal{F} \Delta_0(\mathcal{X}) \mathcal{F}^{-1}, \quad \mathcal{F} = \exp \mathcal{G}, \quad \mathcal{G} = \frac{i}{2} \theta^{\mu\nu} \mathcal{P}_{\mu} \otimes \mathcal{P}_{\nu}. \quad (2.13)$$

Using the notations defined in the previous footnote, the corresponding n-coproduct  $\Delta_{\theta}^{(n)}$  is given by <sup>4</sup>

$$\Delta_{\theta}^{(n)}(\mathcal{X}) = \mathcal{F}_n \Delta_0^{(n)}(\mathcal{X}) \mathcal{F}_n^{-1}, \quad \mathcal{F}_n \equiv \exp \mathcal{G}^{(n)}. \quad (2.14)$$

Since the deformation of the Poincaré UEA affects only its coproduct, the UIR space  $\mathcal{H}$  and the tensor product space  $\mathcal{H}^{\otimes n}$  are not changed but the action of the algebra on  $\mathcal{H}^{\otimes n}$  is deformed as

$$U_{\theta}^{(n)}(\mathcal{X}) = U^{\otimes n}(\Delta_{\theta}^{(n)}(\mathcal{X})). \quad (2.15)$$

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<sup>4</sup>See the appendix for a proof.

Consequently, the tensor algebra representation  $X_\theta$  of  $\mathcal{X}$  is also deformed as

$$X_\theta = 1 \oplus \mathbf{U}(\mathcal{X}) \bigoplus_{n=2}^{\infty} U_\theta^{(n)}(\mathcal{X}). \quad (2.16)$$

On the other hand, from eq. (2.14) we get

$$U_\theta^{(n)}(\mathcal{X}) = U^{\otimes n}(\mathcal{F}_n) U_0^{(n)}(\mathcal{X}) U^{\otimes n}(\mathcal{F}_n)^{-1}, \quad (2.17)$$

so if we define  $F$  as

$$F = 1 \oplus 1 \bigoplus_{n=2}^{\infty} U^{\otimes n}(\mathcal{F}_n), \quad (2.18)$$

we obtain the similarity relation between  $X_0$  and  $X_\theta$ :

$$X_\theta = F X_0 F^{-1}. \quad (2.19)$$

A similar relation can be in fact established for a general counital 2-cocycle  $\mathcal{F}$  [19].

Contrary to the undeformed case, the actions of the twisted Poincaré UEA do not commute with the flips  $\Pi_{ij}$ . From the similarity relation eq. (2.19) we see that they commute with the twisted flips  $F \Pi_{ij} F^{-1}$  which still have eigenvalues  $\pm 1$  and associated eigenspaces, a bosonic Fock space with the eigenvalue 1 and a fermionic Fock space with the eigenvalue  $-1$ . It follows that the bosonic Fock space and a twisted n-bosons state are given by

$$\mathcal{F}_\theta = \mathcal{S}_\theta \mathcal{T}(\mathcal{H}), \quad |p_1, \dots, p_n\rangle_\theta = \mathcal{S}_\theta |p_1\rangle \otimes \dots \otimes |p_n\rangle, \quad (2.20)$$

where the projector  $\mathcal{S}_\theta$  is the twisted symmetrization map with respect to the twisted flip  $F \Pi_{ij} F^{-1}$  and is related to the undeformed symmetrization map by  $\mathcal{S}_\theta = F \mathcal{S}_0 F^{-1}$ .

Since  $F$  is diagonal when acting on  $|p_1\rangle \otimes \dots \otimes |p_n\rangle$ , we obtain a simple relation between the twisted and the untwisted states as

$$F |p_1, \dots, p_n\rangle_0 = f_{p_1 \dots p_n}^\theta |p_1, \dots, p_n\rangle_\theta, \quad (2.21)$$

where  $f_{p_1 \dots p_n}^\theta$  is the eigenvalue of  $F$  with eigenstate  $|p_1\rangle \otimes \dots \otimes |p_n\rangle$ . Explicitly, it is given by

$$f_{p_1 \dots p_n}^\theta = e^{\frac{i}{2} \sum_{i < j} p_i \theta p_j}, \quad F |p_1\rangle \otimes \dots \otimes |p_n\rangle = f_{p_1 \dots p_n}^\theta |p_1\rangle \otimes \dots \otimes |p_n\rangle. \quad (2.22)$$

The action of the translations on the twisted states are unchanged but the action of the Lorentz transformations is deformed as

$$U_\theta(\Lambda) |p_1, \dots, p_n\rangle_\theta = f_{\Lambda p_1 \dots \Lambda p_n}^\theta f_{p_1 \dots p_n}^{-\theta} |\Lambda p_1, \dots, \Lambda p_n\rangle_\theta. \quad (2.23)$$

It is important to notice that the symmetric states  $F |p_1, \dots, p_n\rangle_0$  do not transform covariantly under the twisted Poincaré transformations. They rather transform as do the untwisted states under the untwisted Poincaré transformations. If we demand that the asymptotic states of the theory transform covariantly with  $\Delta_\theta$  then they should be given

by the states  $|p_1, \dots, p_n\rangle_\theta$ . These in turn will determine the creation and annihilation operators.

Notice from eq. (2.21) that the states  $|p_1, \dots, p_n\rangle_\theta$  are not symmetric, the exchange of  $p_i$  and  $p_j$  multiplies the state by a phase:

$$|p_1, \dots, p_j, \dots, p_i, \dots, p_n\rangle_\theta = e^{ip_i\theta p_j} |p_1, \dots, p_n\rangle_\theta. \quad (2.24)$$

The determination of the creation and annihilation operators will also depend on the scalar product of two states which is also changed by the twisting and is given by

$${}_\theta\langle p_1, \dots, p_n | q_1, \dots, q_n \rangle_\theta = \sum_{\mathcal{P}} f_{p_1 \dots p_n}^\theta f_{q_1 \dots q_n}^{-\theta} \delta(p_1 - q_{\mathcal{P}1}) \cdots \delta(p_n - q_{\mathcal{P}n}). \quad (2.25)$$

### 3. Field operators and the Moyal product

A scalar quantum field is an operator acting on the twisted Fock space and can be expressed in terms of the twisted creation and annihilation operators. A covariant field satisfies

$$\Phi(\Lambda x) = U(\Lambda) \Phi(x) U^{-1}(\Lambda), \quad (3.1)$$

where  $\Lambda$  is a Poincaré transformation. In the untwisted case, the covariance relation (3.1) determines completely a scalar field linear in the creation and annihilation operators (see, for instance, section 2 of [20] for a short proof). The resulting scalar field reads

$$\Phi_0(x) = \phi_0^{(+)}(x) + \phi_0^{(-)}(x), \quad \phi_0^{(+)}(x) = \int d\mu(p) e^{ip \cdot x} a_0^\dagger(p), \quad (3.2)$$

where  $\phi^{(-)}$  is the hermitian conjugate of  $\phi^{(+)}$  and  $d\mu(p) = d^{d-1}p (2\pi)^{\frac{1-d}{2}} (2\omega(p))^{-\frac{1}{2}}$  is the invariant measure. The creation operators are defined by their action on the multi-particle states as

$$a_0^\dagger(p) |p_1, \dots, p_n\rangle_0 = |p, p_1, \dots, p_n\rangle_0. \quad (3.3)$$

From the symmetry and the scalar product, we obtain the commutation relations:

$$[a_0^\dagger(p), a_0^\dagger(q)] = [a_0(p), a_0(q)] = 0, \quad [a_0(p), a_0^\dagger(q)] = \delta(p - q). \quad (3.4)$$

We now define the twisted creation and annihilation operators in the same way by

$$a_\theta^\dagger(p) |p_1, \dots, p_n\rangle_\theta = |p, p_1, \dots, p_n\rangle_\theta, \quad (3.5)$$

and also from the symmetry and the scalar product of the twisted states we obtain the following deformed version of the commutation relations:

$$\begin{aligned} a_\theta^\dagger(p) a_\theta^\dagger(q) &= e^{-ip\theta q} a_\theta^\dagger(q) a_\theta^\dagger(p), \\ a_\theta(p) a_\theta(q) &= e^{-ip\theta q} a_\theta(q) a_\theta(p), \\ a_\theta(p) a_\theta^\dagger(q) &= e^{ip\theta q} a_\theta^\dagger(q) a_\theta(p) + \delta(p - q). \end{aligned} \quad (3.6)$$

The relation between the twisted and untwisted states (2.21) can be written in terms of the above defined creation and annihilation operators:

$$\begin{aligned}
 f_{p_1 \dots p_n}^\theta |p_1, \dots, p_n\rangle_\theta &= f_{p_1 \dots p_n}^\theta a_\theta^\dagger(p_1) \cdots a_\theta^\dagger(p_n) |\Omega\rangle \\
 &= \left( e^{\frac{i}{2} p_1 \theta P} a_\theta^\dagger(p_1) \right) \cdots \left( e^{\frac{i}{2} p_n \theta P} a_\theta^\dagger(p_n) \right) |\Omega\rangle \\
 &= F |p_1, \dots, p_n\rangle_0 \\
 &= \left( F a_0^\dagger(p_1) F^{-1} \right) \cdots \left( F a_0^\dagger(p_n) F^{-1} \right) |\Omega\rangle.
 \end{aligned} \tag{3.7}$$

where  $P^\mu = P_\theta^\mu = P_0^\mu$  is the Fock representation of the momentum generator  $\mathcal{P}^\mu$ . From the above equation, we obtain a simple relation between the twisted and untwisted creation operators as

$$F a_0^\dagger(p) F^{-1} = e^{\frac{i}{2} p \theta P} a_\theta^\dagger(p). \tag{3.8}$$

From the unitary transformation of untwisted creation operator by a finite Lorentz transformation  $\Lambda$ :

$$U_0(\Lambda) a_0^\dagger(p) U_0(\Lambda)^{-1} = a_0^\dagger(\Lambda p), \tag{3.9}$$

we obtain the unitary transformation of its twisted counterpart as

$$\begin{aligned}
 U_\theta(\Lambda) \left( e^{\frac{i}{2} p \theta P} a_\theta^\dagger(p) \right) U_\theta(\Lambda)^{-1} &= e^{\frac{i}{2} (\Lambda p) \theta P} a_\theta^\dagger(\Lambda p), \\
 U_\theta(\Lambda) a_\theta^\dagger(p) U_\theta(\Lambda)^{-1} &= e^{\frac{i}{2} (\Lambda p) \theta P - \frac{i}{2} p \theta (\Lambda P)} a_\theta^\dagger(\Lambda p).
 \end{aligned} \tag{3.10}$$

At this stage, we can compare the above defined n-boson states and creation and annihilation operators with those of previous works. The twisted n-particle state considered by Balachandran et al [16] have the same statistics as our states  $|p_1, \dots, p_n\rangle_\theta$ ; the relation between the creation operators  $a_0^\dagger$  and  $a_\theta^\dagger$  differs however by a unitary transformation  $F$ . The states considered by Fiore and Schupp [19], are totally symmetric and correspond to  $F |p_1, \dots, p_n\rangle_0$ . The relation between the twisted and untwisted creation and annihilation operators is obtained by Fiore [21] for general twisted Lie algebra.

Having determined the creation and annihilation operators we can now turn to the construction of the field operator. The scalar field which was used in previous studies [13, 18] is given by

$$\Phi_{\text{nc}\theta}(x) = \phi_{\text{nc}\theta}^{(+)}(x) + \phi_{\text{nc}\theta}^{(-)}(x), \quad \phi_{\text{nc}\theta}^{(+)}(x) = \int d\mu(p) e^{ip \cdot x} a_\theta^\dagger(p). \tag{3.11}$$

and is obtained from  $\Phi_0$  by the replacement of  $a_0^\dagger$  by  $a_\theta^\dagger$ . The so defined field  $\Phi_{\text{nc}\theta}(x)$  is however not covariant. Using eq. (3.10), we deduce the action of a Lorentz transformation:

$$\begin{aligned}
 U_\theta(\Lambda) \Phi_{\text{nc}\theta}(x) U_\theta(\Lambda)^{-1} &= \int d\mu(p) e^{ip \cdot (\Lambda x)} e^{\frac{i}{2} p \theta P - \frac{i}{2} (\Lambda^{-1} p) \theta (\Lambda P)} a_\theta^\dagger(p) \\
 &= \Phi_{\text{nc}\theta}(\Lambda x + \frac{1}{2}(\theta - \Lambda \theta \Lambda)P).
 \end{aligned} \tag{3.12}$$

In fact, we can see from eq. (3.10) that there is no covariant field which is linear in the creation and annihilation operators. If we loosen up the linearity condition, we can get one. In order to see this, let us first define  $\tilde{a}_\theta^\dagger(p)$  by

$$\tilde{a}_\theta^\dagger(p) = e^{\frac{i}{2} p \theta P} a_\theta^\dagger(p) = e^{\frac{i}{2} \int d\mu(q) p \theta q} a_\theta^\dagger(q) a_\theta(q) a_\theta^\dagger(p). \tag{3.13}$$



From the transformation law (3.10), we have

$$U_\theta(\Lambda) \tilde{a}_\theta^\dagger(p) U_\theta(\Lambda)^{-1} = \tilde{a}_\theta^\dagger(\Lambda p), \quad (3.14)$$

and from eq. (3.8), it satisfies the usual commutation relations:

$$[\tilde{a}_\theta^\dagger(p), \tilde{a}_\theta^\dagger(q)] = [\tilde{a}_\theta(p), \tilde{a}_\theta(q)] = 0, \quad [\tilde{a}_\theta(p), \tilde{a}_\theta^\dagger(q)] = \delta(p - q). \quad (3.15)$$

The states generated by successive applications of  $\tilde{a}_\theta^\dagger(p)$  on  $|\Omega\rangle$  are  $F|p_1, \dots, p_n\rangle_0$ . We deduce that the field operator given by

$$\Phi_\theta(x) = \phi_\theta^{(+)}(x) + \phi_\theta^{(-)}(x), \quad \phi_\theta^{(+)}(x) = \int d\mu(p) e^{ip \cdot x} \tilde{a}_\theta^\dagger(p). \quad (3.16)$$

is covariant under the Poincaré transformations. Furthermore, from eq. (3.8) it is related to the untwisted one by a unitary transformation:

$$F \Phi_0(x) F^{-1} = \Phi_\theta(x). \quad (3.17)$$

Therefore, since  $F$  leaves the vacuum state invariant, the QFT with the covariant field  $\Phi_\theta$  (3.16) based on twisted Poincaré UEA leads to the same n-point functions as the untwisted one. This covariant field  $\Phi_\theta$ , which was not considered in the literature before, allows to construct local interaction Hamiltonian's which are manifestly invariant under the twisted symmetry. For example the interaction action  $-\int V(\Phi_\theta)$  with  $V$  an arbitrary potential, is manifestly invariant under the Poincaré transformations. Notice that the Moyal product is not needed to achieve the invariance under the twisted Poincaré symmetry when using the covariant field  $\Phi_\theta$ .

We now turn to examine the relation with the Moyal product which was at the origin of this twisted symmetry. To cover the most general case, we shall consider a generalized version of the Moyal product, which gives the  $*$ -product between two fields at different spacetime points, with an arbitrary twist parameter  $\vartheta$  as

$$(f *_\vartheta g)(x, y) = e^{\frac{i}{2} \partial_x \vartheta \partial_y} f(x) g(y). \quad (3.18)$$

The  $\vartheta = 0$  case corresponds to the usual pointwise product. The  $*_\vartheta$ -product of n plane waves is given by

$$e^{ip_1 \cdot x_1} *_\vartheta \dots *_\vartheta e^{ip_n \cdot x_n} = f_{p_1 \dots p_n}^{-\vartheta} e^{ip_1 \cdot x_1} \dots e^{ip_n \cdot x_n}. \quad (3.19)$$

Consider now the field product

$$(\Phi_\theta *_\vartheta \dots *_\vartheta \Phi_\theta)(x_1, \dots, x_n). \quad (3.20)$$

It can be written as  $\sum_{s_i=\pm} (\phi_\theta^{(s_1)} *_\vartheta \dots *_\vartheta \phi_\theta^{(s_n)})(x_1, \dots, x_n)$  with

$$\begin{aligned} & (\phi_\theta^{(s_1)} *_\vartheta \dots *_\vartheta \phi_\theta^{(s_n)})(x_1, \dots, x_n) = \\ & = \int d\mu(p_1) \dots d\mu(p_n) e^{s_1 i p_1 \cdot x_1} *_\vartheta \dots *_\vartheta e^{s_n i p_n \cdot x_n} \tilde{a}_\theta^{(s_1)}(p_1) \dots \tilde{a}_\theta^{(s_n)}(p_n), \end{aligned} \quad (3.21)$$

where  $\tilde{a}_\theta^{(+)} = \tilde{a}_\theta^\dagger$  and  $\tilde{a}_\theta^{(-)} = \tilde{a}_\theta$ . Using eq. (3.19) and eq. (3.13), the integrand can be rewritten as

$$\begin{aligned}
 & e^{s_1 i p_1 \cdot x_1} *_{\vartheta} \dots *_{\vartheta} e^{s_n i p_n \cdot x_n} \tilde{a}_\theta^{(s_1)}(p_1) \dots \tilde{a}_\theta^{(s_n)}(p_n) \\
 &= e^{s_1 i p_1 \cdot x_1} \dots e^{s_n i p_n \cdot x_n} f_{s_1 p_1 \dots s_n p_n}^{-\vartheta} \left( e^{\frac{i}{2} s_1 p_1 \theta P} a_\theta^{(s_1)}(p_1) \right) \dots \left( e^{\frac{i}{2} s_n p_n \theta P} a_\theta^{(s_n)}(p_n) \right) \\
 &= e^{s_1 i p_1 \cdot x_1} \dots e^{s_n i p_n \cdot x_n} f_{s_1 p_1 \dots s_n p_n}^{-\vartheta + \theta} a_\theta^{(s_1)}(p_1) \dots a_\theta^{(s_n)}(p_n) e^{\frac{i}{2} (s_1 p_1 + \dots + s_n p_n) \theta P} \\
 &= e^{s_1 i p_1 \cdot x_1} *_{\vartheta - \theta} \dots *_{\vartheta - \theta} e^{s_n i p_n \cdot x_n} a_\theta^{(s_1)}(p_1) \dots a_\theta^{(s_n)}(p_n) e^{\frac{i}{2} (s_1 p_1 + \dots + s_n p_n) \theta P}. \quad (3.22)
 \end{aligned}$$

Taking  $x_1 = x_2 = \dots = x_n = x$  in eq. (3.20), using eq. (3.22) and integrating over  $x$  gives

$$\int dx \Phi_\theta *_{\vartheta} \dots *_{\vartheta} \Phi_\theta(x) = \int dx \Phi_{\text{nc}\theta} *_{\vartheta - \theta} \dots *_{\vartheta - \theta} \Phi_{\text{nc}\theta}(x). \quad (3.23)$$

From the similarity relation (3.17), we can also cast the above integral in the form

$$F \left( \int dx \Phi_0 *_{\vartheta} \dots *_{\vartheta} \Phi_0(x) \right) F^{-1}. \quad (3.24)$$

This shows that the invariant interaction given by a monomial potential  $\int dx \Phi_\theta^n(x)$  can also be written as  $\int dx \Phi_{\text{nc}\theta} *_{-\theta} \dots *_{-\theta} \Phi_{\text{nc}\theta}(x)$  or as  $F \left( \int dx \Phi_0^n(x) \right) F^{-1}$ .

Acting with the field product (3.20) on the vacuum state, the last factor in eq. (3.22) drops out and we get

$$\begin{aligned}
 (\Phi_\theta *_{\vartheta} \dots *_{\vartheta} \Phi_\theta)(x_1, \dots, x_n) |\Omega\rangle &= (\Phi_{\text{nc}\theta} *_{\vartheta - \theta} \dots *_{\vartheta - \theta} \Phi_{\text{nc}\theta})(x_1, \dots, x_n) |\Omega\rangle \\
 &= F (\Phi_0 *_{\vartheta} \dots *_{\vartheta} \Phi_0)(x_1, \dots, x_n) |\Omega\rangle, \quad (3.25)
 \end{aligned}$$

where we used eq. (3.17) to get the last equality. This shows that the n-point functions with different choices of fields and products are related to each other in a simple way. The case  $\vartheta = \theta$  shows that using  $\Phi_0$  with the Moyal product, which defines NCFT in the Moyal plane, is equivalent to using the non-covariant field  $\Phi_{\text{nc}\theta}$  with the usual product and is not invariant under the twisted symmetry. The case  $\vartheta = 0$  leads to covariant results and shows that using  $\Phi_\theta$  with the usual product is equivalent to  $\Phi_{\text{nc}\theta}$  with a twisted product but is also related to the untwisted  $\Phi_0$  with the usual product. This agrees with the conclusions of [17, 22, 18].

The theory is fully determined by the asymptotic states and the  $S$ -matrix. We argued that the asymptotic states which transform covariantly under the twisted symmetry are  $|p_1, \dots, p_n\rangle_\theta = f_{p_1 \dots p_n}^{-\theta} F |p_1, \dots, p_n\rangle_0$ . On the other hand the  $S$ -matrix should commute with the transformations  $U_\theta(\mathcal{X})$ . This can be easily realized with an interaction Hamiltonian local in  $\Phi_\theta$ . Since  $\Phi_\theta = F \Phi_0 F^{-1}$ , the obtained  $S$ -matrix is unitarily equivalent to the corresponding untwisted  $S$ -matrix  $S_0$ :  $S = F S_0 F^{-1}$ . The resulting matrix elements are related by phases as

$$\theta \langle q_1, \dots, q_n | S | p_1, \dots, p_m \rangle_\theta = f_{q_1, \dots, q_n}^\theta f_{p_1, \dots, p_m}^{-\theta} \langle q_1, \dots, q_n | S_0 | p_1, \dots, p_m \rangle_0. \quad (3.26)$$

This is the main consequence of the invariance under the twisted Poincaré symmetry.

Our conclusions apply to the twist given by eq. (1.2), other twists of the Poincaré are also possible [23], it would be interesting to explore the consequences of the invariance under these twisted symmetries.

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## A. Twisted n-coproduct

In this appendix we prove eq. (2.14) by induction. First, we have

$$\Delta_{\theta}^{(2)}(\mathcal{X}) = \Delta_{\theta}(\mathcal{X}) = \mathcal{F} \Delta_0(\mathcal{X}) \mathcal{F}^{-1} = \mathcal{F}_2 \Delta_0^{(2)}(\mathcal{X}) \mathcal{F}_2^{-1}. \quad (\text{A.1})$$

Assuming  $\Delta_{\theta}^{(n)}(\mathcal{X}) = \mathcal{F}_n \Delta_0^{(n)}(\mathcal{X}) \mathcal{F}_n^{-1}$ , the next order can be calculated as

$$\begin{aligned} \Delta_{\theta}^{(n+1)}(\mathcal{X}) &= (\Delta_{\theta} \otimes \text{id}^{\otimes(n-1)})(\Delta_{\theta}^{(n)}(\mathcal{X})) \\ &= e^{\mathcal{G}_{12}^{(n+1)}} (\Delta_0 \otimes \text{id}^{\otimes(n-1)})(\mathcal{F}_n \Delta_0^{(n)}(\mathcal{X}) \mathcal{F}_n^{-1}) e^{-\mathcal{G}_{12}^{(n+1)}} \\ &= e^{\mathcal{G}_{12}^{(n+1)}} (\Delta_0 \otimes \text{id}^{\otimes(n-1)})(\mathcal{F}_n) \Delta_0^{(n+1)}(\mathcal{X}) (\Delta_0 \otimes \text{id}^{\otimes(n-1)})(\mathcal{F}_n^{-1}) e^{-\mathcal{G}_{12}^{(n+1)}}. \end{aligned} \quad (\text{A.2})$$

Since  $\Delta_0 \otimes \text{id}^{\otimes(n-1)}$  is linear, we get

$$\begin{aligned} (\Delta_0 \otimes \text{id}^{\otimes(n-1)})(\mathcal{F}_n) &= (\Delta_0 \otimes \text{id}^{\otimes(n-1)}) \left( \exp \left[ \sum_{1 \leq i < j \leq n} \mathcal{G}_{ij}^{(n)} \right] \right) \\ &= \exp \left[ \sum_{1 \leq i < j \leq n} (\Delta_0 \otimes \text{id}^{\otimes(n-1)})(\mathcal{G}_{ij}^{(n)}) \right] \\ &= \exp \left[ \sum_{2 \leq j \leq n} (\Delta_0 \otimes \text{id}^{\otimes(n-1)})(\mathcal{G}_{1j}^{(n)}) + \sum_{3 \leq i < j \leq n+1} \mathcal{G}_{ij}^{(n+1)} \right] \\ &= \exp \left[ \sum_{3 \leq j \leq n} (\mathcal{G}_{1j}^{(n+1)} + \mathcal{G}_{2j}^{(n+1)}) + \sum_{3 \leq i < j \leq n+1} \mathcal{G}_{ij}^{(n+1)} \right]. \end{aligned} \quad (\text{A.3})$$

Finally we obtain

$$e^{\mathcal{G}_{12}^{(n+1)}} (\Delta_0 \otimes \text{id}^{\otimes(n-1)})(\mathcal{F}_n) = \exp \left[ \sum_{1 \leq i < j \leq n+1} \mathcal{G}_{ij}^{(n+1)} \right] = \mathcal{F}_{n+1}, \quad (\text{A.4})$$

and  $\Delta_{\theta}^{(n+1)}(\mathcal{X}) = \mathcal{F}_{n+1} \Delta_0^{(n+1)}(\mathcal{X}) \mathcal{F}_{n+1}^{-1}$ .

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